QUANTUM PERIODS – I. SEMI-INFINITE VARIATIONS OF HODGE STRUCTURES.

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Abstract. We introduce a generalization of variations of Hodge structures living over moduli spaces of non-commutative deformations of complex manifolds. Hodge structure associated with a point of such moduli space is an element of Sato type grassmanian of semi-infinite subspaces in $H^*(X,\mathbb{C})[[\hbar^{-1},\hbar]]$. Periods associated with such semi-infinite Hodge structures serve in order to extend mirror symmetry relations in dimensions greater then three.

Contents

1. Introduction

This paper is concerned with theory of periods of non-commutative varieties which arise naturally in the context of mirror symmetry.

We have noticed in [B1] that in order to understand mirror symmetry phenomena in dimensions higher then three one has to enlarge the class of functions provided by periods of complex manifolds. In this paper we introduce a new class which consists of periods associated with *non-commutative* deformations of complex manifolds. We show that this is precisely the class which is needed in order to generalize to dimensions higher then three the mirror symmetry relation between rational Gromov-Witten invariants of Calabi-Yau manifolds and periods associated with their mirror partners.

As another motivation for developing theory of quantum periods let us mention that such a theory should serve as one of guides for future theory of non-commutative varieties. It is interesting to make a somewhat ambitious parallel and to note that, historically, developement of integration theory of analytic functions

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(theory of elliptic integrals, in particular, and, more generally, of Abelian integrals) was one of the principal reasons for introduction of Riemann surfaces.

One may hope also that quantum periods should help to understand certain aspects of varieties from ordinary commutative world (see, for example, the local Torelli property of the period map from §6.3) and, in particular, especially those related with $K-$theory.

We plan to write a series of papers concerning the theory of quantum periods and its applications. Next paper ([B2]) is devoted to applications to non-commutative geometry and homological mirror symmetry. In a separate paper ([B3]) we consider in details an oversimplified situation where periods of somewhat related type appear. There we deal with semi-infinite Hodge structure associated with mirror partners of projective spaces. The main result of [B3] is the equality expressing generating functions for total sets of rational Gromov-Witten invariants of $\mathbb{CP}^n$in terms of periods of semi-infinite Hodge structures associated with their mirrors.

Let us say a few words on the plan of the paper. Let X denotes a complex smooth manifold. We begin in §2 by recalling description of the moduli space $\mathcal{M}$ of non-commutative deformations of X via differential graded Lie algebra $\mathfrak{g}_{A_\infty}$ of infinitesimal $A_\infty-$symmetries of X. The moduli space $\mathcal{M}$ is the quotient of graded scheme of zeroes of solutions to Maurer-Cartan equation in $\mathfrak{g}_{A_\infty}$ with respect to gauge equivalences. In §3 we show that Gauss-Manin local system extends naturally over the moduli space $\mathcal{M}$. This is based on the general observation (prop.3.2) that sheaves with flat connection over moduli space associated with differential graded Lie algebra $\mathfrak{g}$ can be described via modules over $\mathfrak{g}\otimes\mathbb{C}[\xi]/\xi^2$. In §4 we associate with moduli space $\mathcal{M}$ semi-infinite analog of variations of Hodge structures. If $\gamma\in(\mathfrak{g}_{A_\infty}\otimes\mathfrak{M}_R)^1$ denotes a solution to Maurer-Cartan equation describing a non-commutative deformation of X given by $\phi\in\mathcal{M}(R)$ then the semi-infinite analog of Hodge filtration associated with ϕ is the subspace $L(\phi)\in Gr_{\frac{\infty}{2}}(H[[\hbar^{-1},\hbar]])(R)$ which consists of elements of the form

$$\left[l_\hbar\exp(\frac{1}{\hbar}i_\gamma)(\varphi_0+\varphi_1\hbar+\varphi_2\hbar^2\dots)\right]\text{ where }\varphi_i\in\Omega^*(X,\mathbb{C})\otimes R$$

where i_γ denotes contraction $\Lambda^*T\otimes\Omega^*\to\Omega^*$extended by wedge multiplication by $(0,*)-$forms and $l_\hbar$ denotes rescaling operator $\varphi^{p,q}\to\hbar^{\frac{q-p}{2}}\varphi^{p,q}$ for $\varphi^{p,q}\in\Omega^{p,q}$. In §5 we study variations of the subspaces $L(\phi)$, $\phi\in\mathcal{M}(R)$ and prove that they satisfy semi-infinite version of Griffiths transversality condition:

$$\nabla L\subset\hbar^{-1}L$$

In §6 we explain the role played by periods of such semi-infinite Hodge structures in higher dimensional mirror symmetry. Similar to the three-dimensional case ([COGP]) the calculation of Gromov-Witten invariants of an $n-$dimensional Calabi-Yau manifold Y via quantum periods associated with mirror Calabi-Yau manifold X is done in three steps. At the first step one specifies normalization of the periods. Assume that we are given a linear subspace $S\subset H[[\hbar^{-1},\hbar]]$, $\hbar^{-1}S\subset S$ transversal to subspace $L(0)$ and an element $\psi_0\in L(0)$. Such linear subspace $S=L_W$ and element ψ_0 are associated naturally with monodromy weight filtration W of maximal unipotency cusp on moduli space of complex structures on X. Then there exists unique element $\Psi^W(t,\hbar)\in L(t)$, $t\in\mathcal{M}$ satisfying the normalization condition $\Psi^W(t,\hbar)\in(L_W+\psi_0)$. Second step is a choice of coordinates on the moduli space $\mathcal{M}$ using the periods normalized in the first step. The coordinates are induced via

map to linear space $\mathcal{M} \to L_W/\hbar^{-1}L_W$ defined as $(\Psi^W(t,\hbar) - \psi_0) \bmod \hbar^{-1}L_W$. In the last step one writes differential equation:

$$\frac{\partial^2 \Psi^{W,\alpha}}{\partial t^a_W \partial t^b_W} = \hbar^{-1} \sum_c A^c_{ab}(t_W) \frac{\partial \Psi^{W,\alpha}}{\partial t^c_W}$$

satisfied by quantum periods $\Psi^{W,\alpha}(t_W) = \int_{\Delta_\alpha} \Psi^W$ where $\{\Delta_\alpha\}$ is a basis in $\oplus_k H_k(X,\mathbb{Z})$ and $\{t^a_W\}$ are the coordinates from step two. Then $A^c_{ab}(t_W)$ should coincide with structure constants of (big) quantum cohomology of Y and the coefficients of appropriate Taylor expansion of $A^c_{ab}(t_W)$ should give the set of all rational Gromov-Witten invariants of Y. We checked this in [B1] §6 for Calabi-Yau complete intersections varieties.

2. Moduli spaces of $A_\infty-$deformations of complex manifolds.

It is explained here how to think about moduli space $\mathcal{M}$ of non-commutative deformations of complex manifold X. We assume that the reader is familiar with the technique of deformation theory (see for example, [K2] §4, [B1] §2, and references therein). Idea of considering $A_\infty-$deformations of $D^bCoh(X)$ was suggested in [K1]. The definition of the moduli space given below appeared in [BK].

We would like to stress that a satisfactory definition of objects that are parameterized by moduli space $\mathcal{M}$ remains yet a mystery. The only thing which is known is that it should be some quantum varieties such that their equivalence classes are in one-to-one correspondence with equivalence classes of $A_\infty-$deformations of bounded derived category of coherent sheaves on X. Remark that, as it follows easily from the definition of $\mathcal{M}$ given below, our moduli space contains a closed subspace which parameterizes classes of equivalences of Poisson brackets on X. The space of equivalence classes of first-order deformations parameterized by $\mathcal{M}$ is given by $T_{[X]}\mathcal{M} = \oplus_{p,q} H^q(X, \Lambda^p T_X)[p-q]$. The subspace of Poisson brackets corresponds on the level of first-order deformations to the subspace $H^0(X, \Lambda^2 T_X) \subset (\oplus_{p,q} H^q(X, \Lambda^p T_X)[p-q])$.

2.1. **Deformations of complex structure.**

Recall first the standard description of deformations of complex structures. Given complex manifold X let J denotes the corresponding complex structure on the underlying $C^\infty-$manifold X_{C^∞}. According to Kodaira-Spencer theory the deformations of J are described by elements $\rho \in \Omega^{0,1}(X,T)$ satisfying

$$\overline{\partial}\rho + \frac{1}{2}[\rho, \rho] = 0 \tag{2.1}$$

Namely, to an element ρ one can associate a deformation of the decomposition $T^*_{\mathbb{C}} = T^* \bigoplus \overline{T}^*$ of bundle of $\mathbb{C}-$valued $1-$forms into the sum of sub-bundles corresponding to differentials of holomorphic and anti-holomorphic coordinates. The deformed sub-bundle T^* is given by the graph of the linear map corresponding to $\rho \in \Gamma(X, T \bigotimes \overline{T}^*)$. Eq. (2.1) means that this sub-bundle defines formally integrable distribution. Such a distribution gives a new complex structure on X_{C^∞} by Newlander-Nirenberg theorem. Also, equivalent deformations are related via the action of the Lie algebra $\Gamma_{C^\infty}(X,T)$. Notice that this description is formulated intrinsically in terms of of differential graded Lie algebra on $\oplus_q \Omega^{0,q}(X,T)[-q]$. We denote this differential graded Lie algebra via $\mathfrak{g}_{cs}$.

2.2. $A_\infty-$**deformations.** The $A_\infty-$deformations of X allow similar description using elements $\gamma\in\Omega^{0,*}(X,\Lambda^*T)$. Consider the differential graded Lie algebra

$$\mathfrak{g}_{A_\infty}:=\bigoplus_i \mathfrak{g}^i_{A_\infty}[-i],\ \mathfrak{g}^i_{A_\infty}=\bigoplus_{q-p+1=i}\Omega^{0,q}(X,\Lambda^pT)$$

equipped with Dolbeault differential and the bracket which is given by Schouten-Nijenhuis bracket on Λ^*T extended using cup-product of anti-holomorphic differentials. Consider the graded scheme of solutions to Maurer-Cartan equation

$$MC_{A_\infty}:=\{\overline{\partial}\gamma+\frac{1}{2}[\gamma,\gamma]=0\}\subset\mathfrak{g}_{A_\infty}[1]$$

The standard approach consists of developing solutions to Maurer-Cartan in power series $\gamma=\sum_a\gamma_a t^a+\frac{1}{2}\sum_{a_1a_2}\gamma_{a_1a_2}t^{a_1}t^{a_2}+\ldots$. In this paper we leave aside the question of convergence of power series representing the solutions, which can be treated separately using the standard technique of Kuranishi spaces. Consequently, we will work on the level of formal power series and will be dealing with the completion of MC_{A_∞} at zero. In the sequel MC_{A_∞} will always denote the corresponding formal scheme. This scheme has points defined over arbitrary graded $\mathbb{C}-$Artin algebra[1] R:

$$MC_{A_\infty}(R)=\{\gamma|\overline{\partial}\gamma+\frac{1}{2}[\gamma,\gamma]=0,\,\gamma\in(\mathfrak{g}_{A_\infty}[1]\otimes\mathfrak{M}_R)^0\}$$

where $\mathfrak{M}_R$ denotes the unique maximal ideal in R. We will also consider points of MC_{A_∞} over projective limits of Artin algebras[2], for example, over algebra of formal power series on some finite-dimensional $\mathbb{Z}-$graded vector space. Algebra Lie $\mathfrak{g}_{A_\infty}$ acts on the scheme MC via gauge transformations (again it should be understand as action in the category of formal graded schemes):

$$\alpha\in\mathfrak{g}_{A_\infty}\to\dot{\gamma}=\overline{\partial}\alpha+[\gamma,\alpha] \tag{2.2}$$

For any Artin algebra R the algebra Lie $(\mathfrak{g}_{A_\infty}\otimes\mathfrak{M}_R)^0$is nilpotent and the associated group $\exp\big((\mathfrak{g}_{A_\infty}\otimes\mathfrak{M}_R)^0\big)$ acts on $MC_{A_\infty}(R)$. Explicitly, the action of an element $\varphi=\exp\alpha\in\exp\big((\mathfrak{g}_{A_\infty}\otimes\mathfrak{M}_R)^0\big)$ is given by

$$\gamma\ \to\ \gamma^\varphi:=\overline{\partial}\varphi\cdot\varphi^{-1}+\varphi\cdot\gamma\cdot\varphi^{-1}=\sum_{n=0}^{\infty}(\frac{(ad\,\alpha)^{n+1}\overline{\partial}\alpha}{(n+1)!}+\frac{(ad\,\alpha)^n\gamma}{n!}) \tag{2.3}$$

Definition. The (formal) moduli space $\mathcal{M}$ is the quotient of $MC_{A_\infty}$by the gauge transformations (2.3).

More precisely this should be understand as follows: $\mathcal{M}$ is described via the functor $Def^{\mathbb{Z}}_{\mathfrak{g}_{A_\infty}}$: $\mathbb{Z}-$graded Artin algebras $\to$sets, which is given by

$$Def^{\mathbb{Z}}_{\mathfrak{g}_{A_\infty}}(R):=MC_{A_\infty}(R)/\exp\big((\mathfrak{g}_{A_\infty}\otimes\mathfrak{M}_R)^0\big) \tag{2.4}$$

Morally the value of this functor $Def^{\mathbb{Z}}_{\mathfrak{g}_{A_\infty}}(R)$ on Artin algebra R is the set of morphisms $(\operatorname{Spec}R,0)\to\mathcal{M}$. The tangent space to $\mathcal{M}$ at the base point (more precisely the space of equivalence classes of first-order deformations) is equal to $Def^{\mathbb{Z}}_{\mathfrak{g}_{A_\infty}}(\mathbb{C}[t]/t^2)$ which gives

$$T_{[X]}\mathcal{M}=\oplus_{p,q}H^q(X,\Lambda^pT)[p-q]$$

[1]Recall that any such algebra is isomorphic to $\mathbb{C}[t_s]_{s\in S}/I$, where S is some finite set and I is an ideal containing sufficiently large power of the maximal ideal $t\mathbb{C}[t_s]$.

[2]Here and below by Artin algebra we mean graded $\mathbb{C}-$Artin algebra if it does not specified otherwise.

In the sequel we will denote via $\mathcal{M}(R)$ the set of "points" of $\mathcal{M}$ over R, i.e. set of equivalence classes of $A_\infty-$deformations depending on parameters from R.

If one restricts oneself to solutions to Maurer-Cartan equation from $MC_{cs}(R) := \{\gamma|\overline{\partial}\gamma + \frac{1}{2}[\gamma,\gamma] = 0,\ \gamma \in \Omega^{0,1}(X,T)\otimes \mathfrak{M}_{R^0}\}$ then it corresponds to deformations of complex structure on X. We denote by $\mathcal{M}^{cs}$ the quotient of MC_{cs} by gauge equivalences (2.2). It is easy to see that such equivalences are generated only by elements from $\mathfrak{g}^0_{cs} = \Gamma_{C^\infty}(X,T)$. For compact complex manifold X the formal moduli space $\mathcal{M}_{cs}$ is isomorphic to the completion at the base point $[X]$ of Kuranishi space which is a mini-versal family of deformations of complex structure on X (Goldman-Millson) (here we assumed for simplicity that $H^0(X,T_X) = 0$, otherwise the statement is slightly more complicated).

The formality theorem from [K2] can be used to show that the moduli space $\mathcal{M}$ parameterizes $A_\infty-$deformations of the category $D^bCoh(X)$. The corresponding arguments are sketched in §3.3 of [B1].

There is a useful point of view on the quotients of type (2.4) (see [K2] §4, [B1] §2). Such quotients can be more generally associated with germs of $Q-$manifolds (or dg-manifolds in different terminology). This is a germ of $\mathbb{Z}-$graded manifold plus a degree one vector field Q satisfying $[Q,Q] = 0$. Any differential graded Lie algebra can be viewed as $Q-$manifold. Namely, vector field

$$Q_{\mathfrak{g}}(\gamma) = d\gamma + \frac{1}{2}[\gamma,\gamma]$$

on $\mathfrak{g}[1]$ satisfies $[Q_{\mathfrak{g}},Q_{\mathfrak{g}}] = 0$. Then the moduli space described by $\mathfrak{g}$ can be viewed as a kind of nonlinear cohomology associated with $Q_{\mathfrak{g}}$. More precisely, it is realized as a quotient

$$\mathcal{M}_{\mathfrak{g}} = (\mathrm{zeros\,of}\,Q)/\mathrm{distribution\,generated\,by\,vector\,fields\,\,of\,the\,form}\,[Q,\,\cdot\,]) \tag{2.5}$$

of the type similar to (2.4). It is often useful to express statements about the quotient in $Q-$equivariant terms on the underlying Q-manifold.

Let us add that it is natural to expect that for complex algebraic manifold X the moduli space $\mathcal{M}$ can be equipped with appropriate algebraic structure i.e. $\mathcal{M}$ must be defined canonically on the level of (a kind of) dg-scheme.

3. Gauss-Manin local system over $\mathcal{M}$.

This section is devoted to an analog of Gauss-Manin local system living over the moduli space of $A_\infty-$deformations of X.

Sheaves over moduli spaces are described by modules over corresponding differential graded Lie algebras. The easiest way to see this is to use the geometric language of $Q-$manifolds. Let $\mathfrak{g}$ denotes differential graded Lie algebra describing moduli space $\mathcal{M}_{\mathfrak{g}}$ as in (2.4). Given dg-module h over $\mathfrak{g}$ with its structure morphisms given by linear maps $d_h : h \to h[1]$, $m : \mathfrak{g}\otimes h \to h$ one can define $Q_{\mathfrak{g}}-$connection $\mathcal{D}_{Q_{\mathfrak{g}}}$ on trivial bundle over $\mathfrak{g}[1]$ with fibers equal to the linear space h. The covariant derivatives of constant sections are given by

$$(\mathcal{D}_{Q_{\mathfrak{g}}}s)(\gamma) := d_h s + m(\gamma\otimes s)\ \mathrm{for}\, s\in h \tag{3.1}$$

Using the Leibnitz rule the covariant derivative $\mathcal{D}_{Q_{\mathfrak{g}}}s$ can be defined for arbitrary section of the trivial bundle. The conditions imposed on linear maps (d_h,m) in order that they define dg-module structure on h can be reformulated simply as $[\mathcal{D}_{Q_{\mathfrak{g}}},\mathcal{D}_{Q_{\mathfrak{g}}}] = 0$. It follows that for $\gamma\in MC_{\mathfrak{g}}(R)$ one has twisted differential acting on the fiber over γ giving rise to complex $(h\otimes R, \mathcal{D}_{Q_{\mathfrak{g}}}|_\gamma)$.

Proposition 3.1. *Given a solution to Maurer Cartan equation $\gamma \in MC_{\mathfrak{g}}(R)$ and an element of the group of gauge transformations $\varphi \in \exp\big((\mathfrak{g}_{A_\infty} \otimes \mathfrak{M}_R)^0\big)$ the cohomology of complexes $(h \otimes R, \mathcal{D}_{Q_{\mathfrak{g}}}|_\gamma)$ and $(h \otimes R, \mathcal{D}_{Q_{\mathfrak{g}}}|_{\gamma^\varphi})$ are canonically isomorphic and the correspondence which associates to $\gamma \in MC_{\mathfrak{g}}(R)$ the cohomology of complex $(h \otimes R, \mathcal{D}_{Q_{\mathfrak{g}}}|_\gamma)$ defines a sheaf over the moduli space $\mathcal{M}_{\mathfrak{g}}$.*

Proof. Morally this means that $Q_{\mathfrak{g}}-$equivariant sheaves descend naturally to the quotient $\mathcal{M}_{\mathfrak{g}}$. To check that the cohomology of complexes $(h \otimes R, \mathcal{D}_{Q_{\mathfrak{g}}}|_\gamma)$ and $(h \otimes R, \mathcal{D}_{Q_{\mathfrak{g}}}|_{\gamma^\varphi})$ are canonically isomorphic it is sufficient to consider infinitesimal gauge transformation $\varphi = \varepsilon\alpha$, $\varepsilon^2 = 0$. Then the map $Id + \varepsilon m(\alpha \bigotimes \cdot)$ gives the needed chain isomorphism. □

Same statement holds true for arbitrary $L_\infty-$modules over $L_\infty-$algebra (in this case the vector field Q and the $Q-$connection $\mathcal{D}_Q$ will be given by formal power series and as usual one should express everything in terms of co-connections on comodules etc.).

Below we will specify this description in the case of the moduli space $\mathcal{M}$ and the sheaf corresponding to the natural extension of the Gauss-Manin local system.

3.1. **Local system on $\mathcal{M}^{cs}$.** Let us give first the description of the Gauss-Manin local system $(\mathcal{H}^{cs}, \nabla)$ over the moduli space $\mathcal{M}^{cs}$. The sheaf $\mathcal{H}^{cs}$ is described by $\mathfrak{g}_{cs}-$module $(\Omega^{*,*}, \overline{\partial} + \partial)$ with $\mathfrak{g}_{cs}-$action $\gamma \in \mathfrak{g}_{cs} \to \mathcal{L}_\gamma$, $\mathcal{L}_\gamma := [\partial, i_\gamma]$. This description proceeds as follows. Let $\gamma \in MC_{cs}(R)$ be a solution to eq.(2.1) and $\phi_\gamma : (Spec\, R, 0) \to (\mathcal{M}^{cs}, [X])$ be the corresponding morphism representing a deformation of complex structures over R. The sheaf $\mathcal{H}^{cs}$ can be described via its inverse images $\phi_\gamma^* \mathcal{H}^{cs}$. The sheaf $\phi_\gamma^* \mathcal{H}^{cs} \to \mathrm{Spec}\, R$ is given by $R-$module defined as the cohomology of the complex

$$(\Omega_R^{*,*},\, d_\gamma) := (\Omega^{*,*} \bigotimes R,\ \overline{\partial} + \partial + [i_\gamma, \partial]) \tag{3.2}$$

here i_γ denotes the operation of contraction with a vector field extended via wedge multiplication by $(0,1)-$form. One can show easily that

$$\phi_\gamma^* \mathcal{H}^{cs} = \bigoplus_w R^w f_*(\Omega^*_{\widetilde{X}/\mathrm{Spec}\, R}, d_{\widetilde{X}/\mathrm{Spec}\, R})$$

where $f : \widetilde{X} \to \mathrm{Spec}\, R$ is the flat family of complex structures on X corresponding to γ. The sheaf $\mathcal{H}^{cs}$ is equipped with flat connection

$$\nabla : \mathcal{H}^{cs} \to \mathcal{H}^{cs} \otimes \Omega^1_{\mathcal{M}^{cs}}, \nabla^2 = 0 \tag{3.3}$$

Given a solution $\gamma \in MC_{cs}(R)$ the covariant derivative along a vector field $v \in Der(R)$ acting on element φ of $\phi_\gamma^* \mathcal{H}^{cs}$ is written as

$$\nabla_v \varphi = \frac{\partial \varphi}{\partial v} + i_{\frac{\partial \gamma}{\partial v}}(\varphi)$$

one can check that this is indeed the Gauss-Manin flat connection arising from the identification of de Rham cohomology with Betti cohomology.

3.2. **Local system on $\mathcal{M}$.** Notice that the $\mathfrak{g}_{cs}-$action on $(\Omega^{*,*}, \overline{\partial} + \partial)$ extends naturally to the $\mathfrak{g}_{A_\infty}-$action $\gamma \to \mathcal{L}_\gamma$, $\mathcal{L}_\gamma := [\partial, i_\gamma]$ where i_γ denotes the operator of (holomorphic) contraction $\Lambda^* T \otimes \Omega^* \to \Omega^*$extended by wedge multiplication by $(0,q)-$form. The description of the sheaf $\mathcal{H}$ over $\mathcal{M}$ follows the same scheme as above. Let us notice that the operator $d_\gamma = \overline{\partial} + \partial + [i_\gamma, \partial]$ has the property $d_\gamma^2 = 0$

for arbitrary element $\gamma \in MC_{A_\infty}(R)$ describing an $A_\infty-$deformation $\phi_\gamma : \mathrm{Spec}\, R \to \mathcal{M}$. Therefore one can associate to such element γ an $R-$module $''\phi_\gamma^* \mathcal{H}''$ equal to the cohomology of the complex $(\Omega_R^{*,*},\, d_\gamma)$. Also given two equivalent solutions γ_1, γ_2 the cohomology of the complexes $(\Omega_R^{*,*},\, d_{\gamma_1})$, $(\Omega_R^{*,*},\, d_{\gamma_2})$ are canonically isomorphic. It follows that the formula (3.2) with $\gamma \in \mathfrak{g}_{A_\infty}$ gives a sheaf $\mathcal{H} \to \mathcal{M}$.

To explain this in more details one has to recall first a definition of a notion of sheaf over formal moduli space described by functor (2.4). We will give a sketch of how one should proceed and leave the details to the reader. First of all let us consider functor $\widetilde{Def}_{\mathfrak{g}}^{\mathbb{Z}}$ which associates to graded $\mathbb{C}-$Artin algebra R the groupoid(=category where all morphisms are isomorphisms) whose objects are in one-to-one correspondence with elements $\gamma \in MC(R)$ and where morphisms $\gamma_1 \to \gamma_2$ are in one-to-one correspondence with elements $\widetilde{\alpha} \in \exp\big((\mathfrak{g} \otimes \mathfrak{M}_R)^0\big)$ transforming γ_1 to γ_2. Notice that the functor $Def_{\mathfrak{g}}^{\mathbb{Z}}$ can be obtained from $\widetilde{Def}_{\mathfrak{g}}^{\mathbb{Z}}$ by taking π_0=the sets of connected components of objects. By definition a sheaf over moduli space described by $Def_{\mathfrak{g}}^{\mathbb{Z}}$ is a functor which associates to a graded $\mathbb{C}-$Artin algebra R a representation of the grouppoid $\widetilde{Def}_{\mathfrak{g}}^{\mathbb{Z}}(R)$ in the category of $R-$modules. In our case it is the representation given by correspondence $\gamma \to \phi_\gamma^* \mathcal{H}$.

The sheaf $\mathcal{H}$ is equipped with flat connection defined by the same formula (3.3). Again formally this should be understand in terms of functors on Artin algebras similar to above.

In the sequel we will need a slight generalization of this situation involving formal parameter $\hbar$. The fiber of the sheaf $\mathcal{H}^\hbar$ over $\phi_\gamma \in \mathcal{M}$ is given by cohomologies of complex

$$(\Omega^{*,*} \otimes \mathbb{C}[[\hbar^{-1}, \hbar]], \overline{\partial} + [\partial, i_\gamma] + \hbar \partial) \tag{3.4}$$

The covariant derivative along vector field $v \in Der(R)$ is given by the formula

$$\nabla_v = \frac{\partial}{\partial v} + \frac{1}{\hbar} i_{\frac{\partial \gamma}{\partial v}} \tag{3.5}$$

3.3. **$\mathcal{D}-$modules over moduli spaces.** As a side remark let us mention that more generally, given a moduli space $\mathcal{M}$ described by a differential graded Lie algebra $\mathfrak{g}$, a sheaf H equipped with flat connection ($\mathcal{D}-$module) over $\mathcal{M}$ can be described using the technique of deformation theory as follows. Let us denote via $\mathbb{C}[\xi]$ the differential graded *commutative* algebra with generator ξ, $\xi^2 = 0$, $\deg \xi = -1$ and the differential $d = \frac{\partial}{\partial \xi}$. Notice that the space $\mathfrak{g} \otimes \mathbb{C}[\xi]$ is naturally a differential graded Lie algebra.

Proposition 3.2. *Given a $\mathfrak{g}-$module h such that $\mathfrak{g}$-module structure on h is lifted to structure of a module over the differential graded Lie algebra $\mathfrak{g} \otimes \mathbb{C}[\xi]$ one has natural flat connection ∇ on the sheaf H over $\mathcal{M}_{\mathfrak{g}}$associated with h according to prop.3.1. The covariant derivative along $v \in Der(R)$ acting on cohomology of $\mathcal{D}_Q|_\gamma$, $\gamma \in MC_{\mathfrak{g}}(R)$ is given by $\nabla_\nu = \frac{\partial}{\partial v} + i_{\frac{\partial \gamma}{\partial v}}$where i_α denotes the action of the element $\xi\alpha \in \varepsilon\mathfrak{g}$.*

Proof. It is straightforward to check that $[\nabla, \mathcal{D}_Q] = 0$, $[\nabla, \nabla] = 0$.

In our case $\mathfrak{g}_{A_\infty}-$module is the complex (3.4) with $\mathfrak{g}_{A_\infty}-$action $\gamma \in \mathfrak{g}_{A_\infty} \to \mathcal{L}_\gamma$, $\mathcal{L}_\gamma := [\partial, i_\gamma]$. The operators $\frac{1}{\hbar} i_\gamma$ for $\varepsilon\gamma \in \varepsilon\mathfrak{g}_{A_\infty}$ define the lifting of this action to $\mathfrak{g}_{A_\infty} \otimes \mathbb{C}[\xi]-$action. $\square$

Remark 3.3. Let us mention that the same result holds true for $L_\infty-$module structure over an $L_\infty-$algebra $\mathcal{G}$ which can be lifted to $L_\infty-$module structure over $\mathcal{G}\otimes\mathbb{C}[\xi]$. The proof is the same with the only modification that $[\nabla,\nabla]=-2[\mathcal{D}_Q,S]$ for some operator S. We see that $L_\infty-$module over $\mathcal{G}\otimes\mathbb{C}[\xi]$ describe higher homotopy generalization of the notion of sheaf with flat connection. There is in fact an infinite sequence of operators $\widehat{\nabla}^0=\mathcal{D}_Q$, $\widehat{\nabla}^1=\nabla$, $\widehat{\nabla}^2=S$, $\widehat{\nabla}^3\ldots$ satisfying higher homotopy identities. The existence of such higher homotopies allows one to define canonically a parallel transport operator acting on the complex (h,D_Q) for the connection which is flat only up to homotopy. The details will appear in [B2].

4. Semi-infinite Hodge structure

In this section we define an analog of Hodge filtration for $A_\infty-$deformations $\phi\in\mathcal{M}$.

4.1. **Grassmanian.** For a $\mathbb{Z}_2-$graded $\mathbb{C}-$vector space $H=(H^{\text{even}},H^{\text{odd}})$, denote via $H[[\hbar^{-1},\hbar]]$ the space of formal series

$$\sum_{i=-\infty}^{+\infty}v_i\hbar^i+\sum_{j=-\infty}^{+\infty}u_j\hbar^{j+\frac{1}{2}},\ v_i\in H^{\text{even}},u_j\in H^{\text{odd}}$$

Denote also

$$H[[\hbar]]=\{\sum_{r\geq 0}a_r\hbar^r\},\ H[[\hbar^{-1}]]=\{\sum_{r<0}a_r\hbar^r\}\subset H[[\hbar^{-1},\hbar]]$$

and let $H((\hbar))$ be the analogous ring of Laurent power series and p_+ be the linear projection $H[[\hbar^{-1},\hbar]]\to H[[\hbar]]$ along $H[[\hbar^{-1}]]$.

Let $\dim_{\mathbb{C}}H<\infty$. Let us introduce grassmanian $Gr_{\frac{\infty}{2}}(H[[\hbar^{-1},\hbar]])$ of semi-infinite subspaces in $H[[\hbar^{-1},\hbar]]$. Firstly it contains grassmanian $Gr_{\frac{\infty}{2}}(H((\hbar)))$ which consists of semi-infinite subspaces in $H((\hbar))$. In fact, as we work in the neighborhood of moduli space of complex structures, we will be dealing only with the points of $Gr_{\frac{\infty}{2}}(H[[\hbar^{-1},\hbar]])$ which lie in the formal neighborhood[3] of $Gr_{\frac{\infty}{2}}(H((\hbar)))$. Under points of $Gr_{\frac{\infty}{2}}(H[[\hbar^{-1},\hbar]])$ over (pro-)Artin algebra R we will always mean in this paper points of such neighborhood. More precisely, a point $L\in Gr_{\frac{\infty}{2}}(H[[\hbar^{-1},\hbar]])(R)$ over an Artin algebra R (i.e. a "family of subspaces" represented by morphism $Spec\,R\to Gr_{\frac{\infty}{2}}(H[[\hbar^{-1},\hbar]]))$ is an $R\otimes\mathbb{C}[[\hbar]]-$submodule $L\subset H((\hbar))\otimes R$ such that

$$L\otimes_{\mathbb{C}[[\hbar]]}\mathbb{C}((\hbar))=H((\hbar))\otimes R$$

A point $L\in Gr_{\frac{\infty}{2}}(H[[\hbar^{-1},\hbar]])(R)$ over a pro-Artin algebra R is a submodule in $H[[\hbar^{-1},\hbar]]\widehat{\otimes}R$ given by projective limit of submodules $(proj)\lim_i L\otimes_R R/\mathfrak{M}_R^i\in Gr_{\frac{\infty}{2}}(H[[\hbar^{-1},\hbar]])(R/\mathfrak{M}_R^i)$ defined before. Such submodules always satisfy conditions $\dim_R\ker(p_+|_L)<\infty$, $\dim_R\,\mathrm{coker}\,(p_+|_L)<\infty$.

Let F be a decreasing filtration on H, $\dim_{\mathbb{C}}H<\infty$. A filtration in the category of $\mathbb{Z}_2-$graded vector spaces is given by a pair of filtrations $F=(F^{even},F^{odd})$ on $H^{even}$and H^{odd} respectively. Assume in addition that F^{even} has indexes taking

[3]In general, for deformations which are not of finite order the subspaces which we study do not belong to $Gr_{\frac{\infty}{2}}(H((\hbar)))$. One can repeat the same story in the analytic setting i.e. on the level of convergent solutions to Maurer-Cartan equation. In this case some type of Segal-Wilson grassmanian should be used. Then subspaces associated analogously with such convergent noncommutative deformations (see below) will not in general lie in $H((\hbar))$ as well.

values in $\mathbb{Z}$ and F^{odd} has indexes taking values in $\frac{1}{2}+\mathbb{Z}$. To shorten the notations $F^{\geq r}$will denote below the subspace $(F^{even})^r$ if $r \in \mathbb{Z}$ and the subspace $(F^{odd})^r$ if $r \in \frac{1}{2}+\mathbb{Z}$. To such filtration one can associate a subspace $L^F \in Gr_{\frac{\infty}{2}}(H[[\hbar^{-1},\hbar]])$:

$$L^F := \text{linear span}_{r\in\mathbb{Z}[\frac{1}{2}]} F^{\geq r}\hbar^{-r}[[\hbar]]$$

The correspondence $F \to L^F$ is injective. In particular one has the subspace which corresponds to Hodge filtration[4] on $H^*(X,\mathbb{C})$

(4.1)

$$F^{\geq r}(\phi) := \bigoplus_{p-q\geq 2r,\, p-q\equiv 2r} H^{p,q},\ H^{p,q} = \{[\varphi] \in H^{p+q}(X,\mathbb{C})[-p-q] | \varphi \in \Omega^{p,q}(\phi)\}$$

which is associated with a deformation of complex structure $\phi \in \mathcal{M}^{cs}$.

4.2. **Semi-infinite Hodge structure.** We claim that there is a subspace $L(\phi) \in Gr_{\frac{\infty}{2}}(H^*(X,\mathbb{C})[[\hbar^{-1},\hbar]])(R)$ associated canonically to any $A_\infty-$deformation $\phi \in \mathcal{M}(R)$. For $\phi \in \mathcal{M}^{cs}(R)$ it coincides with the subspace $L^{F(\phi)}$corresponding to the Hodge filtration (4.1).

The subspace $L(\phi)$ is defined as follows. Let $\gamma \in MC(R)$ describes an $A_\infty-$deformation $\phi \in \mathcal{M}(R)$ where R is an Artin algebra or a projective limit of Artin algebras. To such deformation we associate an $R-$module $L(\phi)$ in $H^*(X,\mathbb{C})[[\hbar^{-1},\hbar]]\otimes R$ representing morphism $\operatorname{Spec} R \to Gr_{\frac{\infty}{2}}(H^*(X,\mathbb{C})[[\hbar^{-1},\hbar]])$. Let $l_\hbar$ be rescaling operator:

$$l_\hbar : \varphi^{p,q} \to \hbar^{\frac{q-p}{2}}\varphi^{p,q}$$

Denote $\widetilde{L}_\gamma \subset \Omega^*(X,\mathbb{C})[[\hbar^{-1},\hbar]] \bigotimes R$ the subspace of elements of the form[5]

$$l_\hbar \exp(\frac{1}{\hbar}i_\gamma)(\varphi_0+\varphi_1\hbar+\varphi_2\hbar^2\dots)\ \text{where}\, \varphi_i \in \Omega^*(X,\mathbb{C})\otimes R \tag{4.2}$$

Proposition 4.1. *The subspace $\widetilde{L}_\gamma$ is a sub-complex of the complex*

$$(\Omega^*(X,\mathbb{C})[[\hbar^{-1},\hbar]]\otimes R,\, \hbar^{\frac{1}{2}}(\overline{\partial}+\partial)) \tag{4.3}$$

Proof.

$$l_\hbar^{-1}\hbar^{\frac{1}{2}}(\overline{\partial}+\partial)l_\hbar = (\overline{\partial}+\hbar\partial) \tag{4.4}$$

$$\begin{aligned} \exp(-\frac{1}{\hbar}i_\gamma)(\overline{\partial}+\hbar\partial)\exp(\frac{1}{\hbar}i_\gamma) &= \overline{\partial}+\hbar\partial+\frac{1}{\hbar}(i_{\overline{\partial}\gamma+\frac{1}{2}[\gamma,\gamma]})+[\partial,i_\gamma] = \\ &= \overline{\partial}+[\partial,i_\gamma]+\hbar\partial \end{aligned} \tag{4.5}$$

□

Notice that the cohomology groups of the complex (4.3) are isomorphic to $H^*(X,\mathbb{C})[[\hbar^{-1},\hbar]]\bigotimes R$.

Definition. The $R-$submodule $L(\phi) \subset H^*(X,\mathbb{C})[[\hbar^{-1},\hbar]]\bigotimes R$ consists of cohomology classes represented by elements from $\widetilde{L}_\gamma$.

[4]we use indexes which are shifted with respect to the standard notations

[5]The action of $\exp(\frac{1}{\hbar}i_\gamma)$ on arbitrary element from $\Omega^*(X,\mathbb{C})\widehat{\bigotimes}k[[\hbar]]$ is well-defined even in the case when R is a *pro*-Artin algebra since $\exp(\frac{1}{\hbar}\gamma)$ belongs to subspace of elements of the form $\sum_{j\geq 0}\gamma_j\ \gamma_j \in \mathfrak{M}_R^j \bigotimes \hbar^{-j}\mathfrak{g}_{A_\infty}\widehat{\otimes}k[[\hbar]]$.

Theorem 4.2. *To an arbitrary $A_\infty-$deformation $\phi \in \mathcal{M}(R)$ there is canonically associated subspace $L(\phi) \in Gr_{\frac{\infty}{2}}(H^*(X,\mathbb{C})[[\hbar^{-1},\hbar]])(R)$ defined by formula (4.2). For $\phi \in \mathcal{M}^{cs}(R)$ this is the subspace $L^{F(\phi)}$corresponding to the Hodge filtration (4.1).*

Proof. We need to check first that $L(\phi) = L(\phi')$ for $\gamma \sim \gamma'$. It is enough to prove this for $\gamma' = \gamma + \varepsilon(\overline{\partial}\alpha + [\gamma,\alpha])$, $\varepsilon^2 = 0$. Applying the standard commutation rules

$$[\mathcal{L}_{\gamma_1}, i_{\gamma_2}] = i_{[\gamma_1,\gamma_2]}, \ \ [i_{\gamma_1}, i_{\gamma_2}] = 0$$

one gets:

$$\exp(\frac{1}{\hbar} i_{\gamma+\varepsilon\overline{\partial}\alpha+[\gamma,\alpha]})(\mathrm{Id} + \varepsilon[\partial, i_\alpha]) = \exp(\frac{1}{\hbar} i_\gamma)(\mathrm{Id} + \varepsilon[\overline{\partial} + [\partial, i_\gamma] + \hbar\partial, \frac{1}{\hbar} i_\alpha]) \tag{4.6}$$

Assume that $\varphi \in \ker(\overline{\partial} + \partial) \cap \widetilde{L}_\gamma$, or equivalently

$$\exp(-\frac{1}{\hbar} i_\gamma) l_\hbar^{-1}\varphi \in \ker(\overline{\partial} + [\partial, i_\gamma] + \hbar\partial) \cap \Omega^*(X,\mathbb{C})\widehat{\otimes}\mathbb{C}[[\hbar]]$$

Applying the operator (4.6) to $\widetilde{\varphi} = \exp(-\frac{1}{\hbar} i_\gamma) l_\hbar^{-1}\varphi$ we get

$$\varphi \pm \varepsilon l_\hbar \exp(\frac{1}{\hbar} i_\gamma)(\overline{\partial} + [\partial, i_\gamma] + \hbar\partial)\frac{1}{\hbar} i_\alpha \widetilde{\varphi} \in \widetilde{L}_{\gamma+\varepsilon(\overline{\partial}\alpha+[\gamma,\alpha])}$$

on the other hand

$$l_\hbar \exp(\frac{1}{\hbar} i_\gamma)(\overline{\partial} + [\partial, i_\gamma] + \hbar\partial) = (\overline{\partial} + \partial)\hbar^{\frac{1}{2}} l_\hbar \exp(\frac{1}{\hbar} i_\gamma)$$

Let $\gamma \in \Omega^{0,1}(X,T) \otimes \mathfrak{M}_{R^0}$ be a solution to Maurer-Cartan equation describing a deformation of complex structure $\phi : \operatorname{Spec} R \to \mathcal{M}^{cs}$. Let $\{z^i\}$ be a set of complex coordinates with respect to the fixed initial complex structure J_0. The differentials of the deformed complex coordinates can be written as $(dz^i)^{\mathrm{new}} = dz^i + \sum_{\overline{j}} \gamma^i_{\overline{j}} d\overline{z}^{\overline{j}}$, where $\gamma = \sum_{i,\overline{j}} \gamma^i_{\overline{j}} d\overline{z}^{\overline{j}} \frac{\partial}{\partial z^i}$. Therefore $F^{\geq r}(\phi) = \exp i_{\gamma_*} F^{\geq r}(J_0)$. Notice also that for $\gamma \in \Omega^{0,1}(X,T) \otimes \mathfrak{M}_{R^0}$ one has $l_\hbar \circ \exp(\frac{1}{\hbar} i_\gamma) = \exp(i_\gamma) \circ l_\hbar$. It follows that $L(\phi) = L^{F(\phi)}$ for $\phi \in \mathcal{M}^{cs}(R)$.

It is easy to see that there exist elements $\alpha^k_j \in \Omega^*(X)$, $k = 1, \dots, \dim H^*(X,\mathbb{C})$, $j = 0, \dots, m(k)$, such that $\overline{\partial}\alpha^k_j = \partial\alpha^k_{j-1}$, $\overline{\partial}\alpha^k_0 = 0$, $\partial\alpha^k_{m(k)} = 0$ and such that $\{\sum_{j=0}^{m(k)} \alpha^k_j\}_{k=1,\dots,\dim H^*(X,\mathbb{C})}$ give a basis in $H^*(X,\mathbb{C})$. Then given a solution to Maurer-Cartan equation $\gamma \in MC_{A_\infty}(R)$ over an Artin algebra R describing $\phi_\gamma \in \mathcal{M}(R)$ one has

$$\hbar^N \exp(-\frac{1}{\hbar} i_\gamma) \sum_{j=0}^{m(k)} \alpha^k_j \hbar^j \in \Omega^*(X)\otimes\mathbb{C}[[\hbar]]$$

for sufficiently large N. Therefore $l_\hbar \sum_{j=0}^{m(k)} \alpha^k_j \hbar^j \in \hbar^{-N}\widetilde{L}_\gamma$ and $L(\phi_\gamma) \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C}((\hbar)) = H^*(X,\mathbb{C})((\hbar)) \otimes R$ ∎

Remark 4.3. The operator $\frac{1}{\hbar} i_\gamma$ is an operator of the flat connection on the sheaf of cohomologies of complexes (3.4) described in the previous section. The operator $l_\hbar$ identifies cohomology of complexes (3.4) and (4.3). We see that $L(\phi)$ is the subspace obtained as a result of parallel transport to the base point $[X] \in \mathcal{M}$ of the subspace $\{[\varphi_0 + \varphi_1\hbar + \varphi_2\hbar^2 \dots]\}$ living over $\phi \in \mathcal{M}$ and identification provided by $l_\hbar$.

Remark 4.4. If X is a compact Kahler manifold then $\overline{\partial}+[\partial, i_\gamma]$ and ∂ satisfy $\partial\overline{\partial}-$lemma. It follows that $L(\phi)$ is a free $R-$module for any $\phi_{[\gamma]}\in\mathcal{M}(R)$.

Remark 4.5. As we already mentioned the moduli space $\mathcal{M}$ can be viewed as a moduli space parameterizing $A_\infty-$deformations of $D^bCoh(X)$. The local system over $\mathcal{M}$ is the local system of periodic cyclic homology of the $A_\infty-$categories and the semi-infinite subspace coincides with negative cyclic homology. Operator $l_\hbar$ comes from natural grading on periodic cyclic homology of $D^bCoh(X)$.

5. Griffiths transversality

In this section we study variations of semi-infinite subspaces introduced in the previous section. Recall that the family of filtrations $F^{\geq r}(\phi)$ for deformations of complex structure $\phi\in\mathcal{M}^{cs}$ satisfies Griffiths transversality condition with respect to the Gauss-Manin connection:

$$\nabla F^{\geq r}\subset F^{\geq r-1}$$

Theorem 5.1. *The covariant derivative of* $L(\phi)\in Gr_{\frac{\infty}{2}}(H^*(X,\mathbb{C})[[\hbar^{-1},\hbar]])(R)$, $\phi\in\mathcal{M}(R)$ *with respect to the Gauss-Manin connection satisfies*

$$\nabla L\subset\hbar^{-1}L \tag{5.1}$$

Proof. Let $\gamma\in MC(R)$ describes an $A_\infty-$deformation $\phi_\gamma\in\mathcal{M}(R)$. Let

$$[\varphi]=[l_\hbar\exp(\frac{1}{\hbar}i_\gamma)(\varphi_0+\varphi_1\hbar+\dots)]\in L(\phi_\gamma)\subset H^*(X,\mathbb{C})[[\hbar^{-1},\hbar]]\otimes R$$

describes a family of elements of the varying semi-infinite subspaces. Then for any vector field $v\in Der(R)$:

$$\begin{aligned}\nabla_v[\varphi]=[l_\hbar\exp(\frac{1}{\hbar}i_\gamma)(\partial_v\varphi_0+\partial_v\varphi_1\hbar+\dots)+\\+\frac{1}{\hbar}l_\hbar\exp(\frac{1}{\hbar}i_\gamma)(i_{\partial_v\gamma}\varphi_0+i_{\partial_v\gamma}\varphi_1\hbar+\dots)]\in\hbar^{-1}L(\phi_\gamma)\end{aligned} \tag{5.2}$$

□

Notice that one has induced map (symbol of the Gauss-Manin connection)

$$T_\phi\mathcal{M}\otimes L(\phi)/\hbar L(\phi)\to\hbar^{-1}L(\phi)/L(\phi) \tag{5.3}$$

In the case of deformations of complex structure $\phi\in\mathcal{M}^{cs}$ this is the standard map

$$\left(\bigoplus_{p,q}H^q(X,\Lambda^pT_X)[2]\right)\otimes\left(\bigoplus_r F^{\geq r}/F^{\geq r-1}\right)\to\left(\bigoplus_r F^{\geq r}/F^{\geq r-1}\right) \tag{5.4}$$

Remark 5.2. Let us consider integrals

$$\varphi^i(t,\hbar)=\int_{\Delta_i}\varphi(t,\hbar)\ \text{ for }\ \varphi(t,\hbar)\in L(t), \tag{5.5}$$

where $\{\Delta_i\}$ is a basis in $H^*(X,\mathbb{Z})$ and $t\in\mathcal{M}(R)$ represents a class of $A_\infty-$deformations of complex projective manifold X. Such integrals satisfy system of differential equations which are generalizations to the case of non-commutative deformations of Picard-Fuchs equations for usual periods. This can be seen as follows. Assume that we are given elements $\varphi_\alpha(0)\in F^{\geq r_\alpha}(0)$,$\varphi_\alpha(0)\notin F^{r_\alpha-1}(0)$ which form a basis in $H^*(X,\mathbb{C})$. Let $W_{\leq r}$ be an increasing $\mathbb{Z}[\frac{1}{2}]-$graded filtration opposite

to filtration $F^{\geq r}(0)$: that is, for any r, $\bigoplus_{i\equiv 2r} H^i(X,\mathbb{C}) = F^{\geq r} \oplus W_{\leq r}$. Then $L_W :=$ linear $\mathrm{span}_{r\in\mathbb{Z}[\frac{1}{2}]} W_{\leq r}\hbar^{-r}[[\hbar^{-1}]]$ is a subspace transversal to $L^{F(0)}$. Let us consider elements $\widetilde{\varphi}_\alpha(t,\hbar) = L(t)\cap(\varphi_\alpha(0)\hbar^{-r_\alpha} + L_W)$. Notice that for $t\in\mathcal{M}^{cs}$, $\widetilde{\varphi}_\alpha(t,\hbar) = \hbar^{-r_\alpha}\varphi_\alpha(t)$, where $\varphi_\alpha(t) = F^{\geq r_\alpha}(t)\cap(\varphi_\alpha(0)+W_{r_\alpha})$. Then one can show (see proof of prop.6.5 below for analogous arguments) that the periods $\widetilde{\varphi}^i_\alpha(t,\hbar)$ satisfy the following system of equations: for any vector field $v\in Der(R)$ one has

$$\frac{\partial\widetilde{\varphi}^i_\alpha}{\partial v} = \hbar^{-1}\sum_\beta \Gamma^\beta_v(t)\widetilde{\varphi}^i_\beta$$

where $\hbar^{-1}\Gamma_v(t)$ is the $1-$form representing Gauss-Manin connection (3.5).

Remark 5.3. One can generalize easily the notion of real polarized variation of Hodge structure to the semi-infinite context. For example, the semi-infinite analog of the standard condition[6]

$$\forall\, p,k \quad F^p \oplus \overline{F}^{k-p} = H^k$$

is given by

$$L(\hbar)\oplus\overline{L}(\overline{\hbar}|_{\overline{\hbar}=\hbar^{-1}}) = H^*(X,\mathbb{C})[[\hbar^{-1},\hbar]]$$

satisfied by subspaces $L(t)$, $t\in\mathcal{M}$. Similarly, one can define semi-infinite analog of polarization form.

6. Quantum periods and counting of rational Gromov-Witten invariants on Calabi-Yau manifolds.

In three dimensions mirror symmetry predictions express rational Gromov-Witten invariants of a Calabi-Yau threefold in terms of variations of Hodge structure associated with the deformations of complex structure on the mirror dual Calabi-Yau threefold (see [COGP]). We explain in this section that the "non-commutative" variations of semi-infinite Hodge structures described above play the same role in higher dimensional mirror symmetry. In this section X denotes compact Kahler manifold with $c_1(T_X)\in Pic(X)$ equals to zero.

6.1. **Moduli spaces of $A_\infty-$deformations of Calabi-Yau manifolds.** In the case when $c_1(T_X)\in Pic(X)$ equals to zero the moduli space $\mathcal{M}$ has especially nice properties. Let us denote

$$\mathbf{H} = \bigoplus_{p,q}\mathbb{C}^{dim_{\mathbb{C}}H^q(X,\Lambda^pT)}[p-q]$$

Denote also via $\mathbb{C}[[t_{\mathbf{H}}]]$ the algebra of formal power series on the graded vector space $\mathbf{H}$. It is convenient to fix some choice of a set $\{t^a\}$ of linear coordinates on $\mathbf{H}$.

Proposition 6.1. *(*[BK]*, lemma 2.1) The deformation functor $Def^{\mathbb{Z}}_{\mathfrak{g}_{A_\infty}}$ describing the $A_\infty-$deformations of X is isomorphic to the functor represented by the pro-Artin algebra $\mathbb{C}[[t_{\mathbf{H}}]]$. Equivalently, there exists mini-versal solution to the Maurer-Cartan equation*

$$\widehat{\gamma}(t) = \sum_a \widehat{\gamma}_a t^a + \frac{1}{2!}\sum_{a_1,a_2}\widehat{\gamma}_{a_1a_2}t^{a_1}t^{a_2} + \ldots \ \in MC(\mathbb{C}[[t_{\mathbf{H}}]])$$

[6]here for reader convinience we use the standard grading on components of Hodge filtration

and the cohomology classes $[\widehat{\gamma}_a]$ *form a basis in the cohomology of the complex* $(\mathfrak{g}_{A_\infty}, \overline{\partial})$.

Remark 6.2. The reader shoud be warned that such solution is by no means unique. A choice of class of gauge equivalence of such solutions is equivalent to choice of system of local coordinates: $(\mathbf{H}, 0) \to (\mathcal{M}, [X])$

6.2. **Period map on $\mathcal{M}^{cs}$.** The condition $c_1(T_X) = 0$ implies that there exists "holomorphic volume form" $\Omega \in \Gamma(X, \Omega^n)$, $n = \dim_{\mathbb{C}} X$, which is non-vanishing at any point of X. Such form is defined up to multiplication by a non-zero constant. Assume now that we have a family $\widetilde{X}_t$ of deformations of complex structure on $\widetilde{X}_0 = X$. If we choose a holomorphic volume form Ω_0 on $\widetilde{X}_0$ and a hyperplane $l \subset H^n(X, \mathbb{C})$ which is transversal to the line $\mathbb{C} \cdot \Omega_0$ then the above mentioned ambiguity in the choice of holomorphic volume element can be fixed using the condition

$$\Omega_t^l - \Omega_0 \in l$$

for all values of the parameter t for which the hyperplane l rests transversal to the line $\mathbb{C} \cdot \Omega_t$. The correspondence $[\widetilde{X}_t] \to [\Omega_t^l]$ defines the period map: $\mathcal{M}^{cs} \to H^n(X, \mathbb{C})$.

6.3. **Period map on $\mathcal{M}$.** In order to define analogous map on the moduli space $\mathcal{M}$ let us fix a choice of subspace in $H^*(X, \mathbb{C})[[\hbar^{-1}, \hbar]]$ transversal to $L^{F(0)}$ where $F(0)$ is the initial Hodge filtration on $H^*(X, \mathbb{C})$. Such subspace can be associated naturally with an increasing $\mathbb{Z}[\frac{1}{2}]-$graded filtration $W = (W_{even}, W_{odd})$ on $H^*(X, \mathbb{C})$ which is opposite to $F(0)$ in the following sense:

$$\forall\, r : \bigoplus_{i \equiv 2r} H^i(X, \mathbb{C}) = F^{\geq r} \oplus W_{\leq r} \tag{6.1}$$

In the applications to mirror symmetry W will be the limiting weight filtration associated with a maximal unipotency cusp on moduli space of complex structures on X. The subspace $L_W \subset H^*(X, \mathbb{C})[[\hbar^{-1}, \hbar]]$ associated with the filtration W is defined in a similar manner as above for the decreasing filtration $F^{\geq r}$ (see section 4.1)

$$L_W := \text{linear span}_{r \in \mathbb{Z}[\frac{1}{2}]} W_{\leq r} \hbar^{-r}[[\hbar^{-1}]]$$

where we assumed as above that $W_{\leq r} := W^{even}_{\leq r}$ for $r \in \mathbb{Z}$, and $W_{\leq r} := W^{odd}_{\leq r}$ for $r \in \frac{1}{2} + \mathbb{Z}$. It follows from the condition (6.1) that L_W is transversal to $L^{F(0)}$:

$$H^*(X, \mathbb{C})[[\hbar^{-1}, \hbar]] = L_W \oplus L^{F(0)} \tag{6.2}$$

Assume now that in addition to filtration W satisfying (6.1) a holomorphic volume element Ω_0 for initial complex structure on X is fixed. Let $\gamma \in MC_{A_\infty}(R)$ represents an $A_\infty-$deformation $\phi \in: \mathcal{M}(R)$. Recall that we denoted via $L(\phi) \in H^*(X, \mathbb{C})[[\hbar^{-1}, \hbar]] \otimes R$ an $R-$submodule representing the family of semi-infinite subspaces.

Proposition 6.3. *The intersection* $L(\phi) \bigcap (\Omega_0 \hbar^{-\frac{n}{2}} + L_W \otimes R)$ *consists of a single element.*

Proof. It follows from the condition (6.1) that L_W is transversal to $L^{F(0)}$. One can show that $R-$module $L(\phi)$ is free (it follows, for example, from [B1], prop. 4.2.1). Therefore $H^*(X, \mathbb{C})[[\hbar^{-1}, \hbar]] \otimes R = L(\phi) \bigoplus (L_W \otimes R)$. □

We will denote the element defined by intersection $L(\phi_\gamma)\bigcap(\Omega_0\hbar^{-\frac{n}{2}}+L_W\otimes R)$ via $\Psi^W(\phi,\hbar)$. For a mini-versal solution $\widehat{\gamma}\in MC(\mathbb{C}[t_{\mathbf{H}}])$ this element coincides with the series $(\Phi^W(t_{\mathbf{H}},\hbar)\vdash\Omega_0+\Omega_0)\hbar^{-\frac{n}{2}}$ from [B1] (here $\Phi^W(t_{\mathbf{H}},\hbar)$ is the series defined in eq. (4.28) after theorem 1 in [B1]). The corresponding morphism

$$\Psi^W(\phi_{\widehat{\gamma}},\hbar):\mathcal{M}\to H^*(X,\mathbb{C})[[\hbar^{-1},\hbar]][n(\mathrm{mod})2]$$

does not depend on the choice of the mini-versal solution $\widehat{\gamma}$.

Proposition 6.4. *([B1] propositions 4.2.4 and 4.2.6) The map $\Psi^W(\phi_{\widehat{\gamma}},\hbar)$ has the following properties:*

a) the restriction $\Pi^W:=\Psi^W(\hbar)|_{\hbar=1}$is well-defined and gives a local isomorphism of formal super-manifolds (local Torelli theorem):

$$\Pi^W:(\mathcal{M},[X])\simeq(H^*(X,\mathbb{C})[n(\mathrm{mod})2],[\Omega_0])$$

b) the restriction $\Pi^W|_{\mathcal{M}^{cs}}$ coincides with the classical period map $[X_t]\to[\Omega_{[X_t]}^{W_{\leq\frac{n}{2}}}]\in H^n(X,\mathbb{C})$ defined with help of the hyperplane $W_{\leq\frac{n}{2}}\bigcap H^n(X,\mathbb{C})$.

Let us consider "value at $\hbar=\infty$" of Ψ^W which is a morphism $\Psi^W(\infty):\mathcal{M}\to Gr\,W$ into associated quotient of the filtration W defined as the composition of Ψ^W with affine map $\Omega_0\hbar^{-\frac{n}{2}}+L_W\to L_W/\hbar^{-1}L_W=\bigoplus_r W_{\leq r}/W_{\leq r-1}$ sending $v=\Omega_0\hbar^{-\frac{n}{2}}+v_{-r}\hbar^{-r}+v_{-r+1}\hbar^{-r+1}+\ldots\in\Omega_0\hbar^{-\frac{n}{2}}+W_{\leq r}\hbar^{-r}[[\hbar^{-1}]]$ to $[v_{-r}]\in W_{\leq r}/W_{\leq r-1}$. This morphism is a local isomorphism as well ([B1], eq.(4.25)) and defines a set of local coordinates on $\mathcal{M}$ which we denote by t_W.

6.4. **Mirror symmetry in higher dimensions.** The map

$$\left(\bigoplus_{p,q}H^q(X,\Lambda^pT_X)[-p-q(\mathrm{mod})2]\right)\otimes[\Omega_0]\to\left(\bigoplus_r F^{\geq r}/F^{\geq r-1}\right)[n(\mathrm{mod})2]$$

given by restriction of the symbol of Gauss-Manin connection (5.4) is an isomorphism. Therefore the set $\{[\frac{\partial\Psi^W(t_{\mathbf{H}},\hbar)}{\partial t^a_{\mathbf{H}}}]\,\mathrm{mod}\,L([\widehat{\gamma}])\in\hbar^{-1}L([\widehat{\gamma}])/L([\widehat{\gamma}])\}$ is a set of free generators of $\left(\hbar^{-1}L([\widehat{\gamma}])\right)/L([\widehat{\gamma}])$. This implies that the quantum periods given by components of the map Ψ^W satisfy a system of differential equations (noncommutative extension of a version of Picard-Fuchs equations):

Proposition 6.5. *The series $\Psi^W(t_W,\hbar)$ satisfy:*

$$\frac{\partial^2\Psi^W}{\partial t^a_W\partial t^b_W}=\hbar^{-1}\sum_c A^c_{ab}(t_W)\frac{\partial\Psi^W}{\partial t^c_W}$$

for some $A^c_{ab}(t_W)\in\mathbb{C}[[t_W]]$ where $\{t_W\}$ is a set of linear coordinates on $Gr\,W$ induced by the map $\Psi^W(\infty)$.

Proof. It follows from theorem 5.1 that

$$\frac{\partial^2\Psi^W}{\partial t^a_{\mathbf{H}}\partial t^b_{\mathbf{H}}}\in\hbar^{-2}L([\widehat{\gamma}])$$

Therefore there exist $\left(A^{(-1)}\right)^c_{ab}(t_{\mathbf{H}})$, $\left(A^{(0)}\right)^c_{ab}(t_{\mathbf{H}})\in\mathbb{C}[[t_{\mathbf{H}}]]$ such that

$$\frac{\partial^2\Psi^W}{\partial t^a_{\mathbf{H}}\partial t^b_{\mathbf{H}}}-\hbar^{-1}\sum_c\left(A^{(-1)}\right)^c_{ab}(t_{\mathbf{H}})\frac{\partial\Psi^W}{\partial t^c_{\mathbf{H}}}-\sum_c\left(A^{(0)}\right)^c_{ab}(t_{\mathbf{H}})\frac{\partial\Psi^W}{\partial t^c_{\mathbf{H}}}\in L([\widehat{\gamma}])$$

On the other hand,

$$\frac{\partial^2\Psi^W}{\partial t^a_{\mathbf{H}}\partial t^b_{\mathbf{H}}}, \frac{\partial\Psi^W}{\partial t^a_{\mathbf{H}}} \in L_W\widehat{\otimes}\mathbb{C}[[t_{\mathbf{H}}]]$$

Therefore

$$\frac{\partial^2\Psi^W}{\partial t^a_{\mathbf{H}}\partial t^b_{\mathbf{H}}} = \hbar^{-1}\sum_c \left(A^{(-1)}\right)^c_{ab}(t_{\mathbf{H}})\frac{\partial\Psi^W}{\partial t^c_{\mathbf{H}}} + \sum_c \left(A^{(0)}\right)^c_{ab}(t_{\mathbf{H}})\frac{\partial\Psi^W}{\partial t^c_{\mathbf{H}}}$$

The coordinates $\{t_W\}$ were chosen so that $\Psi^W \bmod \hbar^{-1}L_W : \mathcal{M} \to L_W/\hbar^{-1}L_W$ is linear in $\{t_W\}$. Therefore in these coordinates $\frac{\partial^2\Psi^W}{\partial t^a_W\partial t^b_W} = 0 \bmod \hbar^{-1}L_W$ and $\left(A^{(0)}\right)^c_{ab} = 0$. □

Corollary 6.6. *One has for one-form* $A = \sum_a A^c_{ab}(t_W)dt^a_W$ *:*

$$dA = 0, \quad [A, A] = 0 \tag{6.3}$$

The map Ψ^W induces from the Poincare pairing a bilinear form on the tangent sheaf of $\mathcal{M}$ ([B1] formula (5.59)). If W is isotropic with respect to the Poincare pairing:

$$(\alpha, \beta) = 0, \text{ for } \alpha \in W_{\leq r}, \beta \in W_{\leq -r+1}$$

then the form induced on $T\mathcal{M}$ via $\Psi^W$is written as $\left\langle \frac{\partial}{\partial t^a_W}, \frac{\partial}{\partial t^b_W}\right\rangle = \eta_{ab}\hbar^{n-2}$, where η_{ab} is graded symmetric and non-degenerate. The series $A^c_{ab}(t_W)$ give structure constants of commutative associative multiplication on the fibers of $T\mathcal{M}$ ($[A, A] = 0$). They define together with η_{ab} a quasi-homogeneous solution to WDVV-equation ([B1] theorem 5.6) on (formal) neighborhood of zero in $Gr\, W$ identified with $\mathcal{M}$.

Remark 6.7. Similarly one can construct solutions to WDVV-equation starting from a of kind of abstract semi-infinite variation of Hodge structure of Calabi-Yau type. More precisely it is the data $(\mathcal{G}, \Omega[\hbar])$ where $\mathcal{G}$ is an $L_\infty-$algebra and $\Omega[\hbar]$ is $\mathbb{C}[\hbar]-$linear $L_\infty-$module over $\mathcal{G}\otimes\mathbb{C}^\hbar[\xi]$, here $\mathbb{C}^\hbar[\xi]$ is the differential graded commutative algebra over $\mathbb{C}[\hbar]$ with generator ξ, $\xi^2 = 0$, $\deg\xi = -1$ and the differential $d = \hbar\frac{\partial}{\partial\xi}$. This data define local system $\mathcal{H}^\hbar$ over the moduli space associated with $\mathcal{G}$ and the family of semi-infinite subspaces $L(t) \subset \mathcal{H}^\hbar(t)$ $t \in \mathcal{M}$, which satisfy analog of Griffiths transversality condition (5.1). We assume that moduli space $\mathcal{M}$ has a smooth subspace $\widetilde{\mathcal{M}}$ such that $L(t)|_{\widetilde{\mathcal{M}}}$ is described by a flat $\mathcal{O}_{\widetilde{\mathcal{M}}}-$ module and that Calabi-Yau type condition holds on cohomology level: there exists $[\Omega_0] \in L(0)/\hbar L(0)$, such that the restriction of symbol of Gauss-Manin connection (5.3) to $[\Omega_0]$ gives an isomorphism $T_0\widetilde{\mathcal{M}} \simeq \hbar^{-1}L(0)/L(0)$. Then to any subspace $S \subset \mathcal{H}^\hbar_0$, $\hbar^{-1}S \subset S$ transversal to $L(0)$ one can associate a solution $A^c_{ab}(t)$ to equations (6.3). Assume in addition that the module $\Omega[\hbar]$ is equipped with non-degenerate invariant pairing $G(\hbar a, b) = -G(a, \hbar b) = \hbar G(a, b)$ which induces pairing on $L(0)$ with values in $\hbar^D\mathbb{C}[[\hbar]]$ for some $D \in \mathbb{Z}$. Then for isotropic S ($G|_{\hbar S} \in \hbar^D\mathbb{C}[[\hbar^{-1}]]$) a flat metrics is induced on $\widetilde{\mathcal{M}}$ compatible with multiplication defined by $A^c_{ab}(t)$, i.e. one gets a solution to WDVV equation.

Remark 6.8. Using the previous remark one can repeat the above construction of solutions to WDVV-equation starting from differential graded Lie algebra of Hochcshild cochains describing $A_\infty-$deformations of $A_\infty-$category $D^bCoh(X)$ and the dg-module describing variations of semi-infinite Hodge structure associated with

local system of periodic cyclic homology. Consequently, the above formal power series $A^c_{ab}(t_W)$ can be constructed entirely in terms of $A_\infty-$category $D^bCoh(X)$.

Rational Gromov-Witten invariants of Calabi-Yau manifold Y are encoded in the series $C^c_{ab}(t_H,q)^Y$ of structure constants of (big) quantum cohomology (see [KM]). In [B1] we made a conjecture which can be now reformulated using the fact that the power series Π^W introduced above coincides with analogous power series from [B1]. The conjecture states that rational Gromov-Witten invariants of Y coincide with Taylor coefficients in the series

$$\sum_\alpha \frac{\partial^2 \Pi^{W,\alpha}}{\partial t^a_W \partial t^b_W} \left((\partial \Pi^W)^{-1}\right)^c_\alpha = A^c_{ab}(t_W)^X$$

where $\Pi^{W,\alpha}(t_W) = \int_{\Delta_\alpha} \Pi^W$ are the quantum periods considered as functions on the moduli space $\mathcal{M}$ associated with the mirror dual family of Calabi-Yau manifolds X. Some care is needed here: technically we consider family of expansions of $A^c_{ab}(t_W,q)^X$ with varying base $[X_q] \in \mathcal{M}^{cs}$, $[X_q]$ is close to point of maximal unipotent monodromy on the boundary of moduli space of complex structures on X. This is in agreement with the fact that $C^c_{ab}(t_H,q)^Y$ are formal power series with coefficients in semi-group ring $Q[B]$ where B is the semi-group of algebraic cycles on Y modulo numerical equivalences. Interested reader is referred to [B1] §6 for details. Explicitly, the conjecture now states that the identity

$$C^c_{ab}(t_H,q)^Y = A^c_{ab}(t_W,q)^X \tag{6.4}$$

should hold, where W is the limiting weight filtration on $H^*(X,\mathbb{C})$ associated with a maximal unipotency cusp on moduli space of complex structures on X and appropriate choice of identification of affine spaces $H^*(Y,\mathbb{C}) = Gr\,W$ is assumed. We checked in [B1] this conjecture for projective complete intersections:

Theorem. *(*[B1] *theorem 6.2) The identity $C^c_{ab}(t_H,q)^Y = A^c_{ab}(t_W,q)^X$ holds for projective complete intersection Calabi-Yau varieties and their mirrors.*

Remark 6.9. One can show using prop. 6.4b that in dimensions three this is equivalent to the standard predictions from [COGP] proven in [G]. In [G] some partial higher dimensional generalization is also proven, which deals with ordinary periods and the subset of rational Gromov-Witten invariants corresponding to small quantum cohomology ring.

6.5. **Quantum periods and mirror symmetry for Fano manifolds.** A somewhat similar type of integrals appears in a different setting, namely as periods associated with $A_\infty-$deformations of a pair (X,f) where X is a quasi-projective variety and $f: X \to \mathbb{A}^1$ is a morphism to affine line. Such objects appear as mirror partners to projective complete intersection Fano manifolds. We consider such a case in [B3]. In fact in this case the situation simplifies drastically. The Maurer-Cartan equation becomes empty, the corresponding moduli space can be identified with the moduli space of deformations of the function f and the periods of the semi-infinite Hodge structures are given simply by oscillating integrals. We prove in loc.cit. an identity analogous to (6.4) which expresses total collection of rational Gromov-Witten invariants of $\mathbb{CP}^n$ in terms of periods of semi-infinite Hodge structure associated with its mirror partner.

References

[B1] S. Barannikov, *Generalized periods and mirror symmetry in higher dimensions*, alg-geom/9903124.

[B2] S. Barannikov, *Quantum periods -II,* in preparation.

[B3] S. Barannikov, *Semi-infinite Hodge structures and mirror symmetry for projective spaces,* Preprint ENS DMA-00-24.

[BK] S. Barannikov, M. Kontsevich, *Frobenius manifolds and formality of Lie algebras of polyvector fields*, Intern. Math. Res. Notices, no.4 (1998), 201-215.

[COGP] P. Candelas, X. de la Ossa, P. Green and L. Parkes, *A pair of Calabi-Yau manifolds as an exactly soluble superconformal field theory*, Nuclear Physics **B359** (1991), 21-74.

[G] A. Givental, *Equivariant Gromov-Witten invariants*, Intern. Math. Res. Notices, 13 (1996), 613-663

[K1] M. Kontsevich, *Homological algebra of mirror symmetry,* Proc.of ICM, Zurich 1994, vol I, Birkhauzer (1995), 120-139.

[K2] M. Kontsevich, *Deformation quantization of Poisson manifolds–I*, q-alg/9709040.

[KM] M. Kontsevich, Yu. Manin, *Gromov-Witten classes, quantum cohomology, and enumerative geometry*, Commun. Math. Phys. **164** (1994), 525-562.

URL: `http://www.dma.ens.fr/~barannik/`